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Asymptotic Analysis of Perturbed Mathematical Programs[1]

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We consider a perturbed mathematical programming problem where both the objective and the constraint functions are analytical in both the underlying decision variables and in the perturbation variable/parameter that is denoted by ϵ. The following question arises: what is the description of the solutions of such a perturbed problem when $\epsilon \to 0$? We demonstrate that, under weak conditions, the solutions of the perturbed problems are obtained as Puiseux series expansions in ϵ. The results are obtained by application of the Remmert–Stein representation theorem for complex analytic varieties. © 2000 Academic Press

1. MOTIVATION

Perturbation analysis of the generic mathematical programming problem

$$\min f(\mathbf{x})$$

subject to

$$h_i(\mathbf{x}) = 0, \qquad i = 1, 2, \ldots, p$$

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$$g_j(\mathbf{x}) \leq 0, \qquad j = 1, 2, \ldots, m, \tag{1}$$

where $f, h_i, g_j : \mathbb{R}^n \to \mathbb{R}$ are continuous, smooth functions, is now a well established branch of the subject (eg., see [1, 2, 4]).

While in many—most desirable—situations it can be shown that the effect of perturbations on the solutions of (1) dissipates harmlessly as the perturbation "tends to 0," it is also well known that there are situations when "small" perturbations can induce "large" effects. The following example taken from Pervozvanskii and Gaitsgory [6] illustrates that even in the "simplest" case of a linear program whose coefficients are perturbed linearly by a single parameter, ϵ, a discontinuity in the optimal solution exists at $\epsilon = 0$,

$$\max[x_2]$$

subject to

$$\begin{aligned} x_1 + x_2 &= 1 \\ (1+\epsilon)x_1 + (1+2\epsilon)x_2 &= 1+\epsilon \\ x_1, x_2 &\geq 0. \end{aligned}$$

In particular, $\mathbf{x}^0 = (0, 1)^T$ is optimal when $\epsilon = 0$, while $\mathbf{x}(\epsilon) \equiv (1, 0)^T$ is optimal for all $\epsilon > 0$, no matter how small. Of course, it was observed that the rank of the coefficient matrix of the above linear program changes at $\epsilon = 0$. Indeed, the coefficient matrix is nonsingular for all $\epsilon > 0$ but is singular at $\epsilon = 0$, and hence it is natural to refer to a perturbation such as the above as a *singular perturbation*. Roughly speaking, the subject of perturbation analysis of mathematical programs is divided naturally into the study of singular or "regular" (that is, not singular) perturbations. It is also easy to see why the former are normally regarded as "bad" perturbations and the latter as "good" (or "natural") perturbations. Indeed, some researchers would probably regard a singularly perturbed mathematical programming problem as an anomaly resulting from poor problem specification, something to be eliminated by specifying the "correct" problem. In fact, techniques have been developed to either remove the singularity or to replace the original problem by an appropriately constructed limiting unperturbed problem (e.g., see [6]). These techniques have the advantage that, often, they lead to an improved (possibly more robust) solution to the original application.

Nonetheless, the perspective adopted in this paper and its successor is that

1. Singular perturbations can, and do, occur naturally in many real applications,

2. both singular and regular perturbations can be treated, in a unified manner, by considering series expansions of solutions of perturbed mathematical programs, and

3. the Puiseux series is the natural mathematical object for studying the asymptotic behaviour of a very large class of perturbed mathematical programming problems.

Concerning the first of the above claims it is sufficient to note that in most business or engineering applications the mathematical program (1) contains parameters whose true values are unknown. For instance, a typical parameter p is replaced by an estimate

$$\hat{p} = p + \epsilon(N),$$

where the error term, $\epsilon(N)$, comes from a statistical procedure used to estimate p and N is the number of observations used in that estimation. In most of the valid statistical procedures $\epsilon(N) \downarrow 0$ as $N \uparrow \infty$, in an appropriate sense. Thus, from a mathematical programming point of view, it is reasonable to suppress the argument N and simply concern ourselves with the effects of $\epsilon \downarrow 0$. However, the main point is that since the true value p is inherently unknowable, it is hard to tell whether the "true problem" is singularly or regularly perturbed. Certainly, for some values of p, singular perturbations can arise very naturally in an essentially "correct" way.

Concerning the second claim, it now follows that it is natural to begin our investigations by studying the perturbed mathematical program

$$\min f(\epsilon, \mathbf{x})$$

subject to

$$\begin{aligned} h_i(\epsilon, \mathbf{x}) &= 0, \qquad i = 1, 2, \ldots, p \\ g_j(\epsilon, \mathbf{x}) &\leq 0, \qquad j = 1, 2, \ldots, m, \end{aligned} \tag{2}$$

where all functions may now depend on the perturbation parameter[2] ϵ, in addition to the original decision variables $x_1, x_2, \ldots, x_n$. We also claim that an essential understanding of the asymptotic behaviour of the solutions as $\epsilon \downarrow 0$ can be gained from determining what type of functions $x_k(\epsilon)$'s are, for each $k = 1, 2, \ldots, n$, and that this applies to both regular and singular perturbations. Of course, an explicit functional form cannot be hoped for, at the level of generality considered below. Consequently, if it were possible to characterise $x_k(\epsilon)$'s in terms of series expansion in appropriate powers

[2]Of course, the restriction to a scalar perturbation parameter is a serious one; however, it constitutes a natural starting point.

of ϵ, that would already provide a lot of insight to the asymptotic behaviour of solutions as $\epsilon \downarrow 0$.

Concerning the claim that Puiseux series are the natural mathematical objects to use in this context, we first observe that the class of Puiseux series

$$G(\epsilon) = \sum_{\nu=K}^{\infty} c_\nu \epsilon^{\nu/M},$$

where M is a positive integer and K is an arbitrary (fixed) integer that includes both Laurent and power series. Furthermore, it is well known (e.g., see Kato [3]) that Puiseux series arise naturally in the study of perturbations of the spectrum of linear operators. Finally, the simple example below illustrates that fractional powers of ϵ can occur frequently when solving the optimality conditions of (1). In particular, when considering the unconstrained minimisation of

$$f(\epsilon, x_1, x_2) = \frac{x_1^4}{4} + \frac{x_2^4}{4} + \frac{\epsilon}{3} x_1^3 x_2 + \epsilon x_1,$$

we observe that the first order condition $\partial f/\partial x_1 = \partial f/\partial x_2 = 0$ requires the solution of simultaneous equations

$$\begin{aligned} x_1^3 + \epsilon x_1^2 x_2 + \epsilon &= 0 \\ x_2^3 + \frac{\epsilon}{3} x_1^3 &= 0. \end{aligned}$$

It is easy to check that the solutions can be expressed in the Puiseux series form:

$$\begin{aligned} x_1(\epsilon) &= -\epsilon^{1/3} - \frac{\epsilon^{5/3}}{3\sqrt[3]{3}} \cdots \\ x_2(\epsilon) &= -\frac{\epsilon^{2/3}}{\sqrt[3]{3}} + \frac{\epsilon^2}{3\sqrt[3]{9}} \cdots . \end{aligned}$$

Finally, we claim that Puiseux series expansions of solutions of (2) can be derived from two very different and yet not unrelated perspectives, namely those of complex analytic varieties and the theory of semi-algebraic sets. Because the techniques used to derive the corresponding expansions are quite different and lead to different insights (geometric versus algebraic), our results are summarised in two papers, each of which can be read independently of the other. This first paper is devoted to the complex analytic perspective.

2. COMPLEX ANALYTIC PERSPECTIVE

The perturbed mathematical program (2) introduced in the previous section can be viewed as a special case of a slightly more general problem,

$$\min_{\mathbf{x}} f(\epsilon, \mathbf{x})$$

subject to

$$(\epsilon, \mathbf{x}) \in \Omega \subset \mathbb{R}^{n+1}, \tag{3}$$

where the feasible region Ω is viewed as a subset of $\mathbb{R}^{n+1}$ rather than $\mathbb{R}^n$ because of the inclusion of the perturbation parameter ϵ, even though the minimisation is with respect to $\mathbf{x}$ only. Since the objective is to characterise solutions $\mathbf{x}$ of (3) as functions of ϵ and since this may involve solving simultaneous equations of a finite number of non-linear functions, it is reasonable to expect that the complex space $\mathbb{C}^{n+1}$ may be the natural space to work in. Of course, at the end of the analysis, we shall consider the intersection of the solution sets with $\mathbb{R}^{n+1}$.

Toward this end we assume that, in $\mathbb{C}^{n+1}$, the most general "feasible region" that we shall consider will be a *complex analytic variety* $W \subset \mathscr{W}$, where $\mathscr{W}$ is some open set in $\mathbb{C}^{n+1}$. Recall (e.g., see Whitney [7]) that W is an analytic variety in $\mathscr{W}$ if for each $p \in W$ there exists a neighbourhood U of p and holomorphic functions $\theta_1, \theta_2, \ldots, \theta_s$ such that $\theta_i(z) = 0$ for all $z \in W \cap U$ and $i = 1, 2, \ldots, s$, and W is closed in $\mathscr{W}$.

We begin by fixing some analytic variety W that we shall view as the extension of the feasible region Ω into $\mathbb{C}^{n+1}$. That is, W contains all the points (η, z) of interest and defines $\Omega = W \cap \mathbb{R}^{n+1}$. We adopt the convention that points in Ω will be denoted by (ϵ, x) rather than (η, z) whenever it is necessary to emphasise that they are real valued. Similarly, we define $W_\eta = \{z \in \mathbb{C}^n \mid (\eta, z) \in W\}$ when $\eta \in \mathbb{C}$, $W_\epsilon = \{z \in \mathbb{C}^n \mid (\epsilon, z) \in W\}$ when $\epsilon \in \mathbb{R}$, and $W_\epsilon \cap \mathbb{R}^n = \{x \in \mathbb{R}^n \mid (\epsilon + 0i, x_1 + 0i, \ldots, x_n + 0i) \in W\}$. Finally, we postulate that our objective function in (3) derives from a holomorphic function $f : \mathscr{W} \to \mathbb{C}$ such that

$$f(\Omega) \subset \mathbb{R}.$$

We can now define the minimisation problem (3) as a minimisation problem with respect to the analytic variety W. That is,

$$\min_{x} f(\epsilon, x)$$

subject to

$$x \in W_\epsilon \cap \mathbb{R}^n \tag{4}$$

for any $\epsilon \in \mathbb{R}$ such that $W_\epsilon \cap \mathbb{R}^n \neq \phi$. In Section 3 we shall return to the important "special" case where the variety W is defined as the solution set (in $\mathbb{C}^{n+1}$) of the perturbed set of constraint functions such as those of the "standard" mathematical program (2). However, in this and the next section, the more abstract problem (4) is the object of our investigations.

It is now possible to define the *solution set* of (4) for any $\epsilon > 0$ as $S_\epsilon = \{x \in W_\epsilon \cap \mathbb{R}^n \mid x \text{ attains the minimum in (4)}\}$ and the corresponding set in $\mathbb{R}^{n+1}$, namely, $\mathsf{S}_\epsilon = \{(\epsilon, x) \in \Omega \mid x \in S_\epsilon\}$.

Next we introduce the field of Puiseux series with real coefficients. The elements of this field are functions $G(\epsilon)$ of the form

$$G(\epsilon) = \sum_{k=K}^{\infty} c_k \epsilon^{k/M}, \tag{5}$$

where K is some integer and M is a positive integer and the real coefficients $\{c_k\}_{k=K}^{\infty}$ are such that the above series converges for all ϵ sufficiently small. Of course, c_k's and hence $G(\epsilon)$ can be vector-valued.

The goal of this paper is to establish that—under weak conditions—there exists a Puiseux series $G(\epsilon)$ such that

$$x(\epsilon) = G(\epsilon) \in S_\epsilon \tag{6}$$

for all $\epsilon > 0$ and sufficiently small. The claimed result is, perhaps, not surprising if one recalls (e.g., see Kato [3, p. 65]) that eigenvalues of a perturbed matrix are expressible as Puiseux series. However, to the best of our knowledge, the claimed characterisation has not appeared in the mathematical programming literature. In the remainder of this section we introduce some of the notation that will be used later on.

For any holomorphic function $g\colon \mathcal{W} \to \mathbb{C}$ we define the *gradient* of $g(\eta, z)$ at $z = (z_1, z_2, \ldots, z_n)$ such that $(\eta, z) \in \mathcal{W}$ by

$$\nabla g(\eta, z) = \left(\frac{\partial g}{\partial z_1}, \frac{\partial g}{\partial z_2}, \ldots, \frac{\partial g}{\partial z_n} \right),$$

where $\partial g/\partial z_i$ is evaluated at (η, z). Similarly, the *Hessian matrix* of $g(\eta, z)$ at z is defined by

$$\nabla^2 g(\eta, z) = \left(\frac{\partial^2 g(\eta, z)}{\partial z_i \, \partial z_j} \right)_{i,j=1}^{n,n}.$$

If $v, v' \in \mathbb{C}^m$, then $v.v'$ is the *holomorphic inner product* of v and v', that is, the plain inner product which does not involve conjugation. Finally, if $E \subset \mathbb{C}^m$ the *orthogonal complement* of E is given by

$$E^\perp = \{v \in \mathbb{C}^m \mid e.v = 0, \forall e \in E\}.$$

3. MINIMISERS AS PUISEUX SERIES

In this section we shall derive the first main result of the paper that was already, informally, introduced in Eq. (6).

3.1. *Main Result for the Abstract Formulation*

ASSUMPTION 3.1. Let (4) and S_ϵ be as defined in Section 1. We assume that there exists $\epsilon_0 > 0$ such that the set

$$\mathsf{S}^0 = \bigcup_{\epsilon \in [0, \epsilon_0]} \mathsf{S}_\epsilon$$

is a compact set.

THEOREM 3.1. *If Assumption* 3.1 *holds, then there exists* $\bar{\epsilon} \in (0, \epsilon_0)$ *and a vector-valued Puiseux series* $G(\epsilon) = (G_1(\epsilon), G_2(\epsilon), \ldots, G_n(\epsilon))$ *with real coefficients such that*

$$x(\epsilon) = G(\epsilon) \in S_\epsilon$$

for every $\epsilon \in (0, \bar{\epsilon})$.

The entire section is devoted to the derivation of this result which states that it is possible to find a path of solutions to (4) behaving as a Puiseux series.

In order to demonstrate Theorem 3.1, we shall repeatedly use the following lemma. Note that this lemma states that existence of Puiseux series expansions of minima over the finitely many sub-varieties implies the existence of a Puiseux series expansion of minima over the entire variety.

LEMMA 3.1. *Suppose that there are analytic sub-varieties* $W_1, W_2, \ldots, W_s \subset W$ *such that*:

1. $(W_1 \cup W_2 \cdots W_s) \cap \mathsf{S}_\epsilon \neq \phi$ *for* $\epsilon > 0$ *and sufficiently small.*
2. *For each* $j = 1, 2, \ldots, s$, *Theorem* 3.1 *holds for* W_j.

It follows that Theorem 3.1 *holds for* W.

Proof. By (2), for each $j = 1, 2, \ldots, s$, there exists $\epsilon^j > 0$ and a Puiseux series $G^j(\epsilon)$ with real coefficients, such that

$$x(\epsilon) = G^j(\epsilon) \in S^j_\epsilon, \qquad \epsilon \in (0, \epsilon^j),$$

where S^j_ϵ is the set of global minimisers of $f(\epsilon, x)$ over W_j.

For any $\epsilon \in (0, \epsilon^j)$, the value of the objective function at a global minimum over $(W_j)_\epsilon \cap \mathbb{R}^n$ can be thought of as a function of ϵ, namely, $H_j(\epsilon) = f(\epsilon, G^j(\epsilon))$. But since $f(\epsilon, x)$ is holomorphic and $G^j(\epsilon)$ is a Puiseux series, it follows that $H_j(\epsilon)$ is a Puiseux series on $\epsilon \in (0, \epsilon^j)$. Now, define a new Puiseux series

$$H = \min_{\preceq}\{H_1, \dots, H_s\},$$

where $H_i \preceq H_j$ if and only if $H_i(\epsilon) \le H_j(\epsilon)$ for all $\epsilon > 0$ and sufficiently small. Without loss of generality assume that for some fixed k,

$$H(\epsilon) = H_k(\epsilon) \le H_l(\epsilon), \qquad l = 1, 2, \dots, s$$

on some neighbourhood $(0, \bar{\epsilon})$. Now, define

$$H^*(\epsilon) = \min_{W_\epsilon \cap \mathbb{R}^n} f(\epsilon, x), \text{ for each } \epsilon > 0.$$

For every $\epsilon > 0$ and sufficiently small we have, from (1), that for j (depending on ϵ)

$$H^*(\epsilon) = H_j(\epsilon) \ge H_k(\epsilon) = H(\epsilon),$$

and since $(W_j)_\epsilon \cap \mathbb{R}^n \subset W_\epsilon \cap \mathbb{R}^n$ for all j, including $j = k$, we also have that

$$H^*(\epsilon) \le H_k(\epsilon) = H(\epsilon).$$

Thus for $\epsilon > 0$ and sufficiently small

$$H_k(\epsilon) = f\Big(\epsilon, G^k(\epsilon)\Big) = \min_{W_\epsilon \cap \mathbb{R}^N} f(\epsilon, x) = H^*(\epsilon).$$

Hence, for $\epsilon > 0$ and sufficiently small

$$x(\epsilon) = G^k(\epsilon) \in S_\epsilon.$$

∎

3.2. *Tangent Cones*

In order to generalise the natural concepts from mathematical programming to the complex domain, we now introduce the notion of a tangent cone at a point (η, z) of the variety $W \subset \mathbb{C}^{n+1}$.

DEFINITION 3.1. Let W, W_η, $W \cap \mathbb{R}^{n+1}$ and $W_\epsilon \cap \mathbb{R}^n$ be defined as in Section 1. Let $c_q \to \infty$ denote any sequence of complex numbers such that $|c_q| \to \infty$.

1. The tangent cone at a point $(\eta, z) \in W$ is defined as the set of limit points

$$T(W, \eta, z) = \{v = \lim_{q\to\infty} c_q[(\eta^q, z^q) - (\eta, z)] \mid (\eta^q, z^q) \\ \to (\eta, z) \text{ and } c_q \to \infty\},$$

where $(\eta^q, z^q) \in W$ for all q.

2. The tangent cone at a point $z \in W_\eta$ is defined as

$$T(W_\eta, z) = \{v = \lim_{q\to\infty} c_q[z^q - z] \mid z^q \to z \text{ and } c_q \to \infty\},$$

where $z^q \in W_\eta$ for all q. Note that $T(W_\eta, z)$ is isomorphic to the set $\widehat{T}(W_\eta, z) = \{(0, z) \mid z \in T(W_\eta, z)\} \subset \mathbb{C}^{n+1}$.

3. Similarly to (1) (resp. (2)), the tangent cone at a point $(\epsilon, x) \in W \cap \mathbb{R}^{n+1}$ (resp. $x \in W_\epsilon \cap \mathbb{R}^n$)is defined as

$$T(W \cap \mathbb{R}^{n+1}, \epsilon, x) = \{v = \lim_{q\to\infty} c_q[(\epsilon^q, x^q) - (\epsilon, x)] \mid (\epsilon^q, x^q) \to (\epsilon, x) \text{ and } c_q \to \infty\},$$

(resp.

$$T(W_\epsilon \cap \mathbb{R}^n, x) = \{v = \lim_{q\to\infty} c_q[x^q - x] \mid x^q \to x \text{ and } c_q \to \infty\}),$$

where $(\epsilon^q, x^q) \in W \cap \mathbb{R}^{n+1}$ for all q (resp. $x^q \in W_\epsilon \cap \mathbb{R}^n$) and c_q is real-valued. As in (2) above, $T(W_\epsilon \cap \mathbb{R}^n, x)$ is isomorphic to the set $\widehat{T}(W_\epsilon \cap \mathbb{R}^n, x) = \{(0, x) \mid x \in T(W_\epsilon, x)\} \subset \mathbb{R}^{n+1}$.

Note that $T(W, \eta, z) \subset \mathbb{C}^{n+1}$ (resp. $T(W \cap \mathbb{R}^{n+1}, \epsilon, x) \subset \mathbb{R}^{n+1}$) and $T(W_\eta, z) \subset \mathbb{C}^n$ (resp. $T(W_\epsilon \cap \mathbb{R}^n, x) \subset \mathbb{R}^n$). Note also that if we think of the variety W as the "feasible set," then $T(W, \eta, z)$ is a natural generalisation of the notion of the set of tangent vectors of differentiable curves passing through a given feasible point on a smooth surface. The question of what constitutes smoothness in this context is addressed next.

3.2.1. *Tangent Spaces at Regular Points*

With the variety W acting as a feasible set, the notion of a smooth point of such a set may be replaced by the notion of a regular point. Recall (e.g., see Whitney [7, p. 44]) that $(\eta, z) \in W$ is called a *regular point* of W, if W is an analytic manifold in a neighbourhood of (η, z). Let W^- be the set of all regular points of W. The set $W^\times = W \backslash W^-$ is the set of *singular points* of W. Not surprisingly, perhaps, at regular points of W it is possible to give a more explicit characterisation of the tangent cone. Note that, at a regular point, the tangent cone is in fact a space that will be called tangent space. First, we introduce the notion of a tangent space $T(W, \eta, z)$ being "vertical" with respect to the η-axis. More precisely $T(W, \eta, z)$ is vertical if

$$T(W, \eta, z) \subset \{0\} \times \mathbb{C}^n.$$

The next proposition states that at a regular point (η, z) either the tangent space is vertical or the tangent space, at z, of the section variety W_η is precisely the section tangent space of the variety W at (η, z).

PROPOSITION 3.1. *Let* (η, z) *be a regular point of* W. *Then either*

(i) $\widehat{T}(W_\eta, z) = T(W, \eta, z) \cap \left(\{0\} \times \mathbb{C}^n\right)$, *or*

(ii) $T(W, \eta, z)$ *is vertical.*

Proof. See the Appendix.

The following corollary assumes that W is self-conjugate; that is, $W = \bar{W}$. Intuitively it is not a big restriction since by replacing W by the new analytic variety $W \cap \bar{W}$, the real feasible region $W \cap \mathbb{R}^{n+1}$ of (4) remains the same. We are now in a situation to obtain additional properties at any point $(\epsilon, x) \in W \cap \mathbb{R}^{n+1}$:

COROLLARY 3.1. (i) *If* $(\epsilon, x) \in W^-$, *that is, is regular, then* $W \cup \mathbb{R}^{n+1}$ *is locally a real analytic manifold at* (ϵ, x).

(ii) *The tangent spaces* $T(W, \epsilon, x)$ *and* $T(W \cap \mathbb{R}^{n+1}, \epsilon, x)$ *are actually spanned by the same family of real vectors.*

(iii) *If* $T(W, \epsilon, x)$ *is non-vertical, then* $T(W \cap \mathbb{R}^{n+1}, \epsilon, x)$ *is non-vertical and* $\widehat{T}(W_\epsilon \cap \mathbb{R}^n, x) = T(W \cap \mathbb{R}^{n+1}, \epsilon, x) \cap \left(\{0\} \times \mathbb{R}^n\right)$.

(iv) *The tangent spaces* $\widehat{T}(W_\epsilon \cap \mathbb{R}^n, x)$ *and* $\widehat{T}(W_\epsilon, x)$ *are spanned by the same family of real vectors.*

Proof. See the Appendix.

3.3. *Singular Points and Irreducible Varieties*

An analytic variety W is *irreducible* if it cannot be written as the union of two proper analytic sub-varieties. In this subsection we shall assume that the variety W that is of interest in the analysis of (4) is irreducible. Up to now, we have characterised the tangent space at a regular point in the neighbourhood of a regular point. The next theorem (see for proof [7, Theorem 7A, Chap. 3]) characterises also the tangent space at a regular point in the neighbourhood of a singular point.

THEOREM 3.2. *Suppose that* W *is irreducible. There exists a family of open subsets* Q_s *of* $\mathbb{C}^{n+1}$ *such that* $W = \bigcup_s [W \cap Q_s]$ *and for each* s *there exists a finite set of analytic functions* $w_1, w_2, \ldots, w_m$ (*possibly dependent on* s) *mapping* Q_s *into* $\mathbb{C}^N$ *and satisfying*:

(i) $w_j(\eta, z) = 0$ *at all singular points* $(\eta, z) \in W^\times \cap Q_s$ *for each* $j = 1, 2, \ldots, m$, *and*

(ii) *at each regular point* $(\eta, z) \in W^- \cap Q_s$ *the tangent space* $T(W, \eta, z)$ *is spanned by* $w_1(\eta, z), \ldots, w_m(\eta, z)$.

Not only singular points but also regular points at which the tangent space is vertical may cause some problems. The above result can be adapted to our optimisation problem (4) with the help of the following refinement of the $W = W^{-} \cup W^{\times}$ partition of the variety W. In particular, let

$$W = W^{*} \cup W^{\#},$$

where

$$W^{*} = W \backslash W^{\#}$$

and

$$W^{\#} = W^{\times} \cup \{(\eta, z) \in W \mid T(W, \eta, z) \subset \{0\} \times \mathbb{C}^{n}\}.$$

Thus $W^{\#}$ consists of all singular points of W plus those regular points at which the tangent space is vertical and, by default, W^{*} consists of the regular points at which the same space is non-vertical.

COROLLARY 3.2. *Let W and the family $\{Q_s\}$ be as in Theorem 3.2. There exists a finite set of analytic functions $u_2, u_3, \ldots, u_m$ mapping Q_s into $\{0\} \times \mathbb{C}^{n}$ and satisfying*:

(i) *If $(\eta, z) \in W^{\#} \cap Q_s$, then*

$$u_j(\eta, z) = 0, \qquad j = 2, 3, \ldots, m,$$

and

(ii) *If $(\eta, z) \in W^{*} \cap Q_s$, then the tangent space $\widehat{T}(W_\eta, z)$ is spanned by $u_2(\eta, z), u_3(\eta, z), \ldots, u_m(\eta, z)$.*

Proof. See the Appendix.

3.4. *Variety of Critical Points*

With respect to the holomorphic objective function $f\colon \mathscr{W} \to \mathbb{C}$ such that $f(\mathscr{W} \cap \mathbb{R}^{n+1}) \subset \mathbb{R}$ we have been, slowly, assembling the tools that will enable us to characterise the set of "critical" points that contains all the minima. In a suitable sense, this characterisation will be a natural generalisation of the elementary $f'(x) = 0$ notion. In $\mathbb{R}^n$, a minimum of f at a "regular" point of a constraint surface has the property that the gradient $\nabla f(\mathbf{x})$ is orthogonal to the tangent space. Of course, the word "regular" in the above, standard, mathematical programming context implies the presence of a "constraint qualification" that in our context is a special case of a point being a regular point of a variety. How may this idea be further adapted to our case? In the reminder of this sub-section assume that

(a) W is self-conjugate, and

(b) W is irreducible.

It now follows that, at optimum, the gradient vector of the objective function of (4) lies in the orthogonal complement of $\widehat{T}^{\perp}(W_{\epsilon}, x)$ provided that (ϵ, x) is a regular point at which the tangent space is non-vertical. We make use of condition (a) that insures the validity of (*iv*) of Corollary 3.1.

COROLLARY 3.3. *If* $(\epsilon, x) \in W^{\star} \cap \mathsf{S}_{\epsilon}$, *then* $\nabla f(\epsilon, x) \in \widehat{T}^{\perp}(W_{\epsilon}, x)$.

Proof. See the Appendix.

Observe that minima may also occur at points that do not satisfy standard constraint qualifications. One of the advantages of our technique is that we are able to take all cases into consideration. Toward this goal we now define the set of critical points with respect to f by

$$W^{+} = W^{\#} \cup \{(\eta, z) \mid \nabla f(\eta, z) \in \widehat{T}^{\perp}(W_{\eta}, z)\}.$$

Thus, W^{+} includes all the singular points where the tangent space $T(W, \eta, z)$ is vertical and all the points where the gradient lies in the orthogonal complement of $\widehat{T}(W_{\eta}, z)$. Here condition (b) is used to apply Corollary 3.2 so that the tangent space, if it exists, is nicely characterised in the neighbourhood of any point.

COROLLARY 3.4. (i) *The set of critical points* W^{+} *is a complex analytic variety.*

(ii) *For any* $\epsilon > 0$ *small enough*, $\mathsf{S}_{\epsilon} \subset W^{+}$.

Proof. See the Appendix.

In addition to conditions (a) and (b), suppose that the following property is satisfied by W:

(c) $W = W^{+}$.

The following proposition will play a crucial role in the proof of the main existence result. Let us denote by ∂S_{ϵ} the boundary of S_{ϵ} in $W_{\epsilon} \cap \mathbb{R}^{n}$. Note that for $\epsilon \in [0, \epsilon_0]$, the boundary ∂S_{ϵ} is a non-empty set by Assumption 3.1. It will be seen from the proposition below that if x is a boundary point of S_{ϵ}, then (ϵ, x) is either singular or such that its tangent space is vertical.

PROPOSITION 3.2. (i) *If* $x \in \partial S_{\epsilon}$ *then* $(\epsilon, x) \in W^{\#}$.

(ii) *For* $\epsilon > 0$ *small enough* $\mathsf{S}_{\epsilon} \cap W^{\#} \neq \emptyset$.

Proof. See the Appendix.

Of course, it is hard to see why our variety W should satisfy (a), (b), and (c) all together. The next section will show that it is possible to restrict our analysis to precisely this situation.

3.5. *Proof of the Main Existence Result*

We are now in a position to prove the main result—Theorem 3.1—stated earlier. First of all, observe that if we are able to show Theorem 3.1 when W is irreducible, then we are done. Otherwise by [7, Theorem 1G, Chap. 3, Sect. 1] we have that $W = \bigcup_{i \in \mathcal{J}} W_i$ where W_i's are the irreducible components of W. Since $W \cap \mathsf{S}^0 \neq \phi$ and since S^0 is compact, there must exist finitely many W_i's, say $W_1, W_2, \ldots, W_s$, satisfying

$$(W_1 \cup W_2 \cup \cdots W_s) \cap \mathsf{S}_\epsilon \neq \phi \tag{7}$$

for all $\epsilon > 0$ and sufficiently small. In view of Lemma 3.1 it is enough to show that Theorem 3.1 holds for each W_j which is irreducible.

Observe now that the irreducible variety W is of constant dimension dim (see [7] Theorem 1I, Chap. 3, Sect. 1). Our argument is inductive, based on $p = \dim W$.

(i) If $p = 1$, the theorem follows immediately from the Remmert–Stein lemma (see [7, Theorem 3A, Chap. 3, Sect. 3) and Lemma 3.1. The variety W has finitely many branches B_j intersecting S^0, each one of which is parameterised by a Puiseux series $G_j(\eta)$, possibly with complex coefficients. Clearly, there is a minimal subset of Puiseux series, let us say $G_1, \ldots, G_r$, such that for any $\epsilon > 0$ small enough there exists $j \in [1, r]$ with

$$(\epsilon, G_j(\epsilon)) \in \mathsf{S}_\epsilon.$$

Hence, for each j, the imaginary part of $G_j(\epsilon)$ vanishes for all $\epsilon > 0$ and sufficiently small. This implies that each G_j, $j \in [1, r]$ has real coefficients. Hence it is possible to mimic the proof of Lemma 3.1 to show that one of these Puiseux series, let us denote it by G, satisfies

$$(\epsilon, G(\epsilon)) \in \mathsf{S}_\epsilon$$

for any $\epsilon > 0$ small enough.

(ii) Suppose that Theorem 3.1 holds for any variety W such that $\dim W \leq p$. Consider the case where $\dim W = p + 1$.

Observe that if W is not self-conjugate, then $W \cap \bar{W}$ is a self-conjugate proper subvariety. Let us now take $W \cap \bar{W}$ instead of W and observe that the set of minimisers of the new problem is the same as in the initial problem since we have kept all the points in the real domain. However,

$W \cap \bar{W}$ might not be irreducible. However, [7, Theorem 1I, Chap. 3, Sect. 1] states, in particular, that a proper subvariety of an irreducible variety, if not irreducible, is decomposable into irreducible subvarieties of strictly lower dimension.

The inductive hypothesis could then be applied to any of those irreducible subvarieties so that, as above, application of Lemma 3.1 yields the required conclusion.

Similarly, observe that if W^+ is a proper subvariety of W, by taking W^+ instead of W the set of minimisers is unchanged by Corollary 3.4. Again W^+ might not be irreducible, but the very same Theorem 1I in Whitney [7] ensures that we can apply the inductive hypothesis and Lemma 3.1 after appropriate decomposition into irreducible sub-varieties to obtain the required conclusion.

Hence there is no loss of generality in considering only the case where all three conditions (a), (b), and (c) are satisfied.

Now, by Proposition 3.2 part (ii) we have that, for $\epsilon > 0$ and sufficiently small, by considering $W^\#$ in place of W, we retain all the minimisers of (4). However, $W^\#$ is always a proper sub-variety of W and hence is of lower dimension than W. The conclusion now follows by the inductive hypothesis.

4. MINIMISATION UNDER CONSTRAINTS

We now return to the original mathematical programming problem (2), but with the simplification that there are only p equality constraints: $h_i(\epsilon, x) = 0,\ i = 1, 2, \ldots, p$. We shall return to the case of both equality and inequality constraints at the end of this section. To cast the problem in a setting similar to that of Sections 2 and 3, we assume that $\mathscr{W}$ is an open set in $\mathbb{C}^{n+1}$ and $h_1, h_2, \ldots, h_p, f$ are all holomorphic functions mapping $\mathscr{W} \to \mathbb{C}$ such that $\mathscr{W} \cap \mathbb{R}^{n+1}$ is mapped by these functions into $\mathbb{R}$. We consider the perturbed minimisation problem

$$\min f(\epsilon, x)$$

subject to

$$h_i(\epsilon, x) = x, \qquad i = 1, 2, \ldots, p. \tag{8}$$

Let $h = (h_1, \ldots, h_p), \mathscr{W} \to \mathbb{C}^p$, and define the set

$$W = h^{-1}(0, \ldots, 0) = \{(\eta, z) \mid h_i(\eta, z) = 0;\ i = 1, 2, \ldots, p\}.$$

Clearly, as the zero set of p holomorphic functions, W is a complex analytic variety. For a fixed η, let

$$\nabla h_i(\eta, z) = \left(\frac{\partial h_i}{\partial z_1}(\eta, z), \ldots, \frac{\partial h_i}{\partial z_n}(\eta, z) \right)$$

for all z such that $(\eta, z) \in W$, and $i = 1, 2, \ldots, p$. Let $\Gamma(\eta, z)$ be the subspace of $\mathbb{C}^n$ spanned by $\nabla h_i(\eta, z)$ for $i = 1, 2, \ldots, p$. We are now ready to generalise a standard "second order optimality condition" to this new setting.

DEFINITION 4.1. We shall say that a point $(\epsilon, x) \in \mathscr{W} \cap \mathbb{R}^{n+1}$ satisfies optimality conditions of the second order (or is a strict *stationary point*) if

(i) the gradients of the constraints are independent; that is, $\dim \Gamma(\epsilon, x) = p$.

(ii) $\nabla f(\epsilon, x) \in \Gamma(\epsilon, x)$; that is, there exist Lagrange multipliers (dependent on ϵ)
$\lambda_1, \lambda_2, \ldots, \lambda_p \in \mathbb{R}$, not all zero, such that

$$\sum_{i=1}^{p} \lambda_i \nabla h_i(\epsilon, x) + \nabla f(\epsilon, x) = 0.$$

(iii) The Hessian of the Lagrangian of (8) is positive definite on $\Gamma^{\perp}(\epsilon, x)$; that is,

$$L(\epsilon, x, \lambda) = \sum_{i=1}^{p} \lambda_i \nabla^2 h_i(\epsilon, x) + \nabla^2 f(\epsilon, x)$$

is a positive definite matrix.

Note that conditions (i)–(iii) are analogous to the standard second order necessary conditions for strict local minimum (e.g., see Luenberger [5]). Let $\mathscr{S}$ denote the set strict stationary points in $\mathscr{W} \cap \mathbb{R}^{n+1}$ and let $\bar{\mathscr{S}}$ be the closure of $\mathscr{S}$.

Our objective in this section is to prove an analog of Theorem 3.1 but for a set of "solutions" of (4) that are stationary points satisfying (i), (ii), and (iii).

Motivated by the Kuhn–Tucker type condition (ii) we shall now consider the subset of the feasible region W defined by

$$W_1 = \{(\epsilon, x) \in W \mid \operatorname{rank}[\nabla h_1(\epsilon, x), \ldots, \nabla h_p(\epsilon, x), \nabla f(\epsilon, x)] \leq p\},$$

where $[\nabla h_1(\cdot), \ldots, \nabla h_p(\cdot), \nabla f(\cdot)]$ is an $n \times (p+1)$ matrix whose columns are the above gradient vectors. Since the rank condition defining W_1 consists of certain determinants being equal to zero, W_1 is clearly a complex analytic variety. Furthermore, since (ii) holds at any $(\epsilon, x) \in S$, we have that $S \subset W_1$.

LEMMA 4.1. *Let $\mathscr{V} \subset \mathscr{W}$ be the open set of points (η, z) satisfying the independent gradients condition (i). Suppose, in addition, that $(\eta, z) \in \mathscr{V} \cap$*

W_1. There exists a unique set of holomorphic functions $\mathscr{V} \to \mathbb{C}$ such that $\lambda_i = \lambda_i(\eta, z)$, $i = 1, \ldots, p$, are the unique Lagrange multipliers satisfying

$$\sum_{i=1}^{p} \lambda_i \nabla h_i(\eta, z) + \nabla f(\eta, z) = 0 \tag{9}$$

for $(\eta, z) \in \mathscr{V} \cap W_1$.

Proof. Multiplying (9) by the transpose of $\nabla h_j(\eta, z)$ for $j = 1, 2, \ldots, p$ yields a set of p equations

$$\sum_{i=1}^{p} [\nabla h_j(\eta, z).\nabla h_i(\eta, z)]\lambda_i = -\nabla h_j(\eta, z)\nabla f(\eta, z)$$

for $j = 1, 2, \ldots, p$. We can think of the above system of equations, with the argument (η, z) suppressed, as simply the linear system

$$A\lambda = b,$$

where the (i, j)th element of A is $a_{ij} = \nabla h_j(\eta, z).\nabla h_i(\eta, z)$ for $i, j = 1, 2, \ldots, p$, and $b_i = -\nabla h_i(\eta, z)\nabla f(\eta, z)$ for $i = 1, 2, \ldots, p$.

It is now easy to check that the independent gradients condition (i) implies that A is nonsingular. Hence $\lambda = A^{-1}b$ defines the unique set Lagrange multiplier solutions $\lambda_i(\eta, z)$, $i = 1.2, \ldots, p$, satisfying (ii). Clearly, these functions are holomorphic.

THEOREM 4.1. *The complex analytic variety W_1 is one-dimensional near any $(\epsilon, x) \in S$.*

Proof. Consider a holomorphic function F: $\mathscr{W} \times \mathbb{C}^p \to \mathbb{C}^{p+n}$ defined by

$$F(\eta, z, \lambda) = \left(h_1(\eta, z), \ldots, h_p(\eta, z), \sum_{i=1}^{p} \lambda_i \nabla h_i(\eta, z) + \nabla f(\eta, z) \right),$$

where $z = (z_1, \ldots, z_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_p)$.

Note that the zero set of F, namely

$$W_2 := F^{-1}(0),$$

is a complex analytic variety in $\mathscr{W} \times \mathbb{C}^p$. Let $A(\eta, z) = (\nabla h_1(\eta, z), \ldots, \nabla h_p(\eta, z))$ be the $n \times p$ matrix of gradients of h_i's and $L(\eta, z, \lambda) = \sum_{i=1}^{p} \lambda_i \nabla^2 h_i(\eta, z) + \nabla^2 f(\eta, z)$ be the Hessian of the Lagrangian as in (iii). Hence for $(\eta, z, \lambda) \in W_2$ the Jacobian of F with respect to (z, λ) is given by the $(p + n) \times (p + n)$ matrix

$$\frac{\partial F}{\partial(z, \lambda)} = \begin{pmatrix} A^T(\eta, z) & 0 \\ L(\eta, z, \lambda) & A(\eta, z) \end{pmatrix}.$$

We claim that $\frac{\partial F}{\partial(z,\lambda)}$, is nonsingular at $(\eta, z, \lambda) = (\epsilon, x, \lambda)$ satisfying (i), (ii) and (iii). To check this suppose that there exists (u, v) (not equal to 0) such that $\left[\frac{\partial F}{\partial(z,\lambda)}\right](u, v)^T = 0$. That is,

$$A^T(\epsilon, x)u^T = 0$$
$$L(\epsilon, x, \lambda)u^T + A(\epsilon, x)v^T = 0.$$

However, the first of the above equations implies that $uA(\epsilon, x) = 0$, so multiplying the second equation by u, on the left, yields

$$uL(\epsilon, x, \lambda)u^T = 0.$$

However, the positive definiteness of $L(\epsilon, x, \lambda)$ implies that $u = 0$ which in turn leads to $A(\epsilon, x)v^T = 0$, which contradicts (i). We can now apply the implicit function theorem to show that in a neighbourhood $U_2(\subset \mathbb{C}^{n+1} \times \mathbb{C}^p)$ of (ϵ, x, λ), $W_2 \cap U_2$ is a one-dimensional manifold. Furthermore, define a map π: $\mathcal{V} \cap W_1 \to W_2$ by

$$\pi(\eta, z) = (\eta, z, \lambda(\eta, z)),$$

where $\mathcal{V}$ is as in Lemma 4.1. For some sufficiently small neighbourhood $U_1(\subset \mathcal{C}^{n+1})$ of (ϵ, x)

$$\pi(W_1 \cap U_1) \subset W_2 \cap U_2.$$

However, since $W_2 \cap U_2$ is a one-dimensional manifold, the z and λ coordinates of $\pi(\eta, z)$ can be parameterised by η via holomorphic functions. That is, $\pi(\eta, z) = (\eta, z(\eta), \lambda(\eta))$ where $\lambda(\eta) = \lambda(\eta, z(\eta))$, $z = z(\eta)$ for $(\eta, z) \in W_1 \cap U_1$. Hence $(\eta, z) = (\eta, z(\eta))$ on $W_1 \cap U_1$ and therefore $W_1 \cap U_1$ is also a one-dimensional manifold.

We are now in a position to state and prove the main theorem of this section.

THEOREM 4.2. *Given any $(0, x) \in \overline{S}$ there exists an n-vector of Puiseux series in ϵ (with real coefficients), $G(\epsilon) = (G_1(\epsilon), G_2(\epsilon), \ldots, G_n(\epsilon))$ such that for $\epsilon > 0$ and sufficiently small*

$$(\epsilon, G(\epsilon)) \in S$$

and

$$G(0) = \lim_{\epsilon \downarrow 0} G(\epsilon) = x.$$

Proof. Let $\overline{Q}$ be a compact neighbourhood of $(0, x)$. By Theorem 4.1 take a sequence $\{(\epsilon_q, x_q)\}_{q=1}^{\infty}$ in $(W_1 \cap \bar{Q}) \cap \mathscr{S}$ such that $\epsilon_q \downarrow 0$ and $x_q \to x$, as $q \to \infty$. Since $\bar{Q}$ is compact, only finitely many of the one-dimensional components of W_1 intersect $\overline{Q}$. By Theorem 4.1, infinitely many of the points (e_q, x_q) must lie in at least one such component. Let $\overline{W}_1$ be such and irreducible, one-dimensional component and assume, without loss of generality, that $\{(e_q, x_q)\}_{q=1}^{\infty} \subset \overline{W}_1$.

Because $\overline{W}_1$ is one-dimensional the Remmert–Stein representation theorem ensures that there exists an n-vector of Puiseux series $G(\epsilon) = (G_1(\epsilon), \ldots, G_n(\epsilon))$ with real coefficients such that for $\epsilon > 0$ and sufficiently small

$$(\epsilon, x') = (\epsilon, G(\epsilon)) \in \overline{W}_1. \tag{10}$$

In particular, for members of the sequence in $\overline{W}_1$

$$x_q = G(\epsilon_q) \to G(0) = x. \tag{11}$$

Note also that while we know that $(\epsilon_q, x_q) = (\epsilon_q, G(\epsilon_q)) \in \mathscr{S}$ for all $q = 1, 2, \ldots,$ we need prove that is also the case for all $\epsilon > 0$ and sufficiently small. That is, we need to verify that (i)–(iii) are satisfied at $(\epsilon, G(\epsilon))$ for all $\epsilon > 0$ and sufficiently small. These can be verified by recalling that for any Puiseux series $H(\epsilon)$, with real coefficients, if a statement $H(\epsilon) = (or \ \geq \ or \ \leq)$ constant is valid for all $\epsilon_q \downarrow 0$, then it is valid for all $\epsilon > 0$ and sufficiently small. This is a consequence of the fact that $H(\epsilon_q) = 0$ for all $\epsilon_q \downarrow 0$ implies $H(\epsilon) = 0$ for all $\epsilon > 0$ and sufficiently small.

Further, since x_q is real for every $q = 1, 2, \ldots,$ we have from (11) that

$$\mathbb{I}_m(G(\epsilon_q)) = 0$$

infinitely often in the neighbourhood of $\epsilon = 0$. Hence $\mathbb{I}_m(G(\epsilon)) \equiv 0$ and $G(\epsilon) \in \mathbb{R}^n$ in that neighbourhood. Now, verification of (i)–(iii) at $(\epsilon, G(\epsilon))$ for $\epsilon > 0$ and sufficiently small becomes a simple matter. For instance, if (i) were not satisfied for such ϵ, then the matrix

$$A(\epsilon) = (a_{ij}(\epsilon))_{i,j=1}^{p}$$

where $a_{ij}(\epsilon) = \nabla h_j(\epsilon, G(\epsilon)).\nabla h_i(\epsilon, G(\epsilon))$ for all $i, j = 1, 2, \ldots, p$ is singular at $\epsilon = \epsilon_q$ for $q = 1, 2, \ldots, \infty$. Thus the Puiseux series $H(\epsilon) := \det[A(\epsilon)] = 0$ for all $\epsilon = \epsilon_q$. Hence $H(\epsilon) \equiv 0$ for all $\epsilon > 0$ and sufficiently small, yielding the desired contradiction. Similarly (ii) and (iii) can be verified. This completes the proof.

Remark. It is easy to check that the results of this section extend naturally to the case where (8) is replaced by

$$\min f(\epsilon, x)$$

subject to

$$h_i(\epsilon, x) = 0, \qquad i = 1, 2, \ldots, p$$

$$g_j(\epsilon, x) \leq 0, \qquad j = 1, 2, \ldots, m.$$

In this case, by considering at each feasible point (ϵ, x) the combined set of equality and "active" inequality constraints, the problem is effectively reduced to (8). Of course, active inequalities are those that are equal to 0 at the point (ϵ, x), in question.

APPENDIX

5.1. *Proof of Proposition* 3.1

Since (η, z) is a regular point we may assume that in some neighbourhood U of that point, the dimension of W is r, where $1 \leq r \leq n+1$. It is now possible to partition the set of coordinates of (η, z) by partitioning the index set $I = \{0, 1, 2, \ldots, n\}$ corresponding to variables $\eta, z_1, z_2, \ldots, z_n$, respectively. In particular since, locally, $W \cap U$ is a manifold of dimension r, there exists a subset $B = \{i_1, i_2, \ldots, i_r\}$ of I of "basic coordinates" such that every variable with an index that is not in B is locally expressible as a holomorphic function of basic coordinates. There are now two cases.

Case A. The variable η is a basic coordinate; that is, $0 \in B$. Without loss of generality assume that $B = \{0, 1, \ldots, r-1\}$. We know that there exist holomorphic functions $\phi_1(\cdot) \cdots \phi_{n-r+1}(\cdot)$ such that for every $j \notin B$

$$z'_j = \phi_{j-r+1}(\eta', z'_1, \ldots, z'_{r-1}) \text{ and } (\eta', z') \in W \cap U.$$

It follows that $T(W, \eta, z)$ is spanned by the vectors

$$v_0 = \left(1, 0, \ldots, 0, \frac{\partial \phi_1(\eta, z)}{\partial \eta}, \ldots, \frac{\partial \phi_{n-r+1}(\eta, z)}{\partial \eta}\right)$$

$$v_1 = \left(0, 1, \ldots, 0, \frac{\partial \phi_1(\eta, z)}{\partial z_1}, \ldots, \frac{\partial \phi_{n-r+1}(\eta, z)}{\partial z_1}\right)$$

$$\vdots$$

$$v_{r-1} = \left(0, 1, \ldots, 1, \frac{\partial \phi_1(\eta, z)}{\partial z_{r-1}}, \ldots, \frac{\partial \phi_{n-r+1}(\eta, z)}{\partial z_{r-1}}\right).$$

Hence, locally, points $z' \in W_\eta$ are of the form

$$z' = \left(z'_1, \ldots, z'_{r-1}, \phi(\eta, z'_1, \ldots, z'_{r-1}), \ldots, \phi_{n-r+1}(\eta, z'_1, \ldots, z'_{r-1})\right)$$

and therefore $\widehat{T}(W_\eta, z)$ is spanned only by the $r-1$ vectors: $v_1, v_2, \ldots, v_{r-1}$. However, the subspace spanned by these vectors is precisely

$$T(W, \eta, z) \cap \{\{0\} \times \mathbb{C}^n\}$$

and hence Case A is covered by part (i) of the proposition.

Case B. The variable η is not a basic coordinate (that is, $0 \notin B$). Without loss of generality assume that $B = \{1, 2, \ldots, r\}$. Hence, there exist holomorphic functions $\psi_1(\cdot), \ldots, \psi_{n-r+1}(\cdot)$ such that

$$\begin{aligned} \eta' &= \psi_1(z'_1, \ldots, z'_r) \\ z'_j &= \psi_{j-r+1}(z'_1, \ldots, z'_r), \qquad j = r+1, r+2, \ldots, n \end{aligned}$$

for $(\eta', z') \in W \cap U$. As before, $T(W, \eta, z)$ is spanned by the r vectors

$$\begin{aligned} u_0 &= \left(\frac{\partial \psi_1(\eta, z)}{\partial z'_1}, 1, 0, \ldots, 0, \frac{\partial \psi_2(\eta, z)}{\partial z'_1}, \ldots, \frac{\partial \psi_{n-r+1}(\eta, z)}{\partial z'_1} \right) \\ u_1 &= \left(\frac{\partial \psi_1(\eta, z)}{\partial z'_2}, 0, 1, \ldots, 0, \frac{\partial \psi_2(\eta, z)}{\partial z'_2}, \ldots, \frac{\partial \psi_{n-r+1}(\eta, z)}{\partial z'_2} \right) \\ &\vdots \\ u_{r-1} &= \left(\frac{\partial \psi_1(\eta, z)}{\partial z'_r}, 0, 0, \ldots, 1, \frac{\partial \psi_2(\eta, z)}{\partial z'_r}, \ldots, \frac{\partial \psi_{n-r+1}(\eta, z)}{\partial z'_r} \right). \end{aligned}$$

There are now two further cases to be considered:

Case B1. This is the case where

$$\frac{\partial \psi_1(\eta, z)}{\partial z'_k} = 0 \text{ for all } k \in B.$$

In this case u_i has a zero in the first entry, for each $i = 0, 1, \ldots, r-1$. Hence,

$$T(W, \eta, z) \subset \{0\} \times \mathbb{C}^n,$$

and part (ii) of the proposition applies.

Case B2. For some $k \in B$, without loss of generality say $k = 1$, $\partial \psi_1 / \partial z'_1 \neq 0$. We want to demonstrate that in this case it is possible to replace B by a new set of basic coordinates $\widehat{B} = \{0, 2, 3, \ldots, r\}$, thereby reducing this to Case A. This is achieved with the help of the analytic implicit function theorem (e.g., Whitney [7, pp. 302–303]).

Note that changing from B to $\widehat{B}$ is equivalent to replacing the basic variable z_1' with η' and keeping $z_2', z_3', \ldots, z_r'$ as still basic. Now define $(n-r+1)$ holomorphic functions on $W \cap U$ as follows:

$$F_1(\eta', z') = \eta' - \psi_1(z_1', \ldots, z_r')$$
$$F_j(\eta', z') = z_{r+j-1}' - \psi_j(z_1', \ldots, z_r'), \qquad j = 2, 3, \ldots, n-r+1.$$

It follows from the definition of ψ_j's that $F_j(\eta', z') = 0$ on $W \cap U$ for each $j = 1, 2, \ldots, n-r+1$. Now partitioning the variables $(\eta', z_1', \ldots, z_n')$ according to $\widehat{B}$ and its complement $\widehat{B}'$ we note that the Jacobian of F_j's with respect to the non-basic variables corresponding to $\widehat{B}'$ is of the form

$$\mathcal{J} = \frac{\partial(F_1, \ldots, F_{n-r+1})}{\partial(z_1', z_{r+1}', \ldots, z_n')} = \begin{pmatrix} \frac{\partial F_1}{\partial z_1'} & \frac{\partial F_2}{\partial z_1'} & \cdots & \frac{\partial F_{n-r+1}}{\partial z_1'} \\ \frac{\partial F_1}{\partial z_{r+1}'} & \frac{\partial F_2}{\partial z_{r+1}'} & \cdots & \frac{\partial F_{n-r+1}}{\partial z_{r+1}'} \\ \vdots & & & \\ \frac{\partial F_1}{\partial z_n'} & \frac{\partial F_2}{\partial z_n'} & \cdots & \frac{\partial F_{n-r+1}}{\partial z_n'} \end{pmatrix}.$$

We can partially evaluate the above Jacobian at (η, z) by noting that due to the construction of F_j's

$$\frac{\partial F_1(\eta, z)}{\partial z_1'} = -\frac{\partial \psi_1(\eta, z)}{\partial z_1'} \neq 0$$

and

$$\frac{\partial F_j(\eta, z)}{\partial z_{r+k}'} = \begin{cases} 0 & \text{if } j = 1 \text{ and } k \geq 1 \\ 1 & \text{if } j \geq 2 \text{ and } k = j-1 \\ 0 & \text{otherwise.} \end{cases}$$

This is sufficient to check that the determinant of the Jacobian evaluated at (η, z) is

$$|\mathcal{J}| = -\frac{\partial \psi(\eta, z)}{\partial z_1'} \cdot 1 \neq 0.$$

Hence $\mathcal{J}$ is invertible and hence it is possible to express all the non-basic variables $z_1', z_{r+1}', z_{r+2}', \ldots, z_n'$ as holomorphic implicit functions of the basic variables $\eta', z_2', \ldots, z_r'$, on $W \cap U$. This case is now equivalent to Case A, and hence part (i) of the proposition applies.

5.2. *Proof of Corollary* 3.1

Since (ϵ, x) is such that $T(W, \epsilon, x)$ is non-vertical, without loss of generality we can assume that we are in Case A of the proof of Proposition 3.1. Hence, regarding (ϵ, x) as a special case of (η, z) we have that in a neighbourhood of (ϵ, x), $\widehat{T}(W_\epsilon, x)$ is spanned by the $r-1$ vectors $v_1, v_2, \ldots, v_{r-1}$.

From the construction of these vectors and the assumption $\bar{W} = W$, we see that for all $(\epsilon, z'_1, \ldots, z'_{r-1}, z'_r, \ldots, z'_n)$ in a neighbourhood of (ϵ, x), $z'_i = \bar{z}'_i = x'_i$ for $i = 1, 2, \ldots, r-1$ and hence

$$z'_j = \phi_{j-r+1}(\epsilon, x'_1, x'_2, \ldots, x'_{r-1}) = \bar{z}'_j = x'_j \tag{12}$$

for $j = r, 2, \ldots, n$. Hence the vectors $v_1, v_2, \ldots, v_{r-1}$ are in fact real-valued in a neighbourhood of (ϵ, x), and span $\widehat{T}(W_\epsilon, x)$. However, it also follows from (12) that, in a neighbourhood of (ϵ, x), the real analytic variety $W_\epsilon \cap \mathbb{R}^n$ is a manifold. Then by the same argument as used in Case A of Proposition 3.1, $v_1, v_2, \ldots, v_{r-1}$ also span $\widehat{T}(W_\epsilon \cap \mathbb{R}^n, x)$.

5.3. *Proof of Corollary* 3.2

If $(\eta, z) \in W^* \cap Q_s$ we can, without loss of generality, assume that we are in Case A of the proof of Proposition 3.1. Let $w_1(\eta, z), w_2(\eta, z), \ldots, w_m(\eta, z)$ be as in Theorem 3.2. Since by Proposition 3.1(i) every point of $\widehat{T}(W_\eta, z)$ is a point of $T(w, \eta, z)$ with a 0 in its first entry, it follows from part (ii) of Theorem 3.2 that

$$\widehat{T}(W_\eta, z) = \left\{ \sum_{j=1}^{m} u_j w_j(\eta, z) \mid \sum_{j=1}^{m} u_j w_j^0(\eta, z) \right.$$
$$\left. = 0, u_j \in \mathbb{C} \text{ for each } j = 1, 2, \ldots, m \right\},$$

where $w_j^0(\cdot)$ is the first entry of $w_j(\eta, z)$ for each $i = 1, 2, \ldots, n$. Define the set of vectors $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ in $\mathbb{C}^m$ corresponding to the above as

$$M_0^\perp = \left\{ \mu \in \mathbb{C}^m \mid \sum_{j=1}^{m} \mu_j w_j^0(\eta, z) = 0 \right\}.$$

Recalling the proof of Case A of Proposition 3.1 we note that there is no loss of generality in assuming that $w_1^0(\eta, z) = 1$. Now, it is easy to check that $M_0^\perp$ is spanned by the vectors $m_j = (-w_j^0(\eta, z), 0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 entry in m_j occurs in the jth place for each $j = 2, \ldots, m$. Clearly the vectors m_j constitute a basis of $M_0^\perp$. Now we can construct $m-1$ analytic functions (mapping Q_s into $\{0\} \times \mathbb{C}^n$) by

$$u_j(\eta, z) = -w_j^0(\eta, z) w_1(\eta, z) + w_1^0(\eta, z) w_j(\eta, z), \tag{13}$$

for each $j = 2, 3, \ldots, m$. Note that the first entry of $u_j(\eta, z)$ is given by

$$u_j^0(\eta, z) = -w_j^0(\eta, z).1 + 1.w_j^0(\eta, z) = 0 \tag{14}$$

for each $j = 2, \ldots, m$. Hence $u_j(\eta, z) \in \{0\} \cap \mathbb{C}^n$ for each $j = 2, \ldots, m$. Since $T(W, \eta, z)$ is spanned by $w_1(\eta, z), \ldots, w_m(\eta, z)$ it now follows that $\widehat{T}(W_\eta, z)$ is spanned by $u_j(\eta, z)$ for $j = 2, 3, \ldots, m$. Hence part (ii) of the corollary holds.

If $(\eta, z) \in W^\times$, then $w_j(\eta, z) = 0$ for $j = 1, 2, \ldots, m$ by Theorem 3.2(i) and $u_j(\eta, z) = 0$ for $j = 2, 3, \ldots, m$ by (13).

If $(\eta, z) \in W^-$ and is such that $T(W, \eta, z)$ is vertical then by Case B1 of Proposition 3.1 every point in $T(W, \eta, z)$ is a linear combination of vectors with 0 in the first entry. Since by Theorem 3.2(ii) $T(W, \eta, z)$ is still spanned by $w_j(\eta, z)$; $j = 1, \ldots, m$, we have that

$$w_j^0(\eta, z) = 0, \quad j = 1, 2, \ldots, m.$$

Of course, construction(13) of u_j's still applies; however, we readily observe that the right side of (13) is now identically 0 for each $j = 2, 3, \ldots, m$. Thus part (i) of the corollary holds again.

5.4. *Proof of Corollary* 3.3

Since $T(W, \epsilon, x)$ is non-vertical, we are in Case A of the proof of Proposition 5. With $v_1, v_2, \ldots, v_{r-1}$ as in the proof of Corollary 3.1 we observe that

$$\nabla f(\epsilon, x) \cdot v_i = \frac{\partial f}{\partial x_i} + \sum_{j=r}^{n} \frac{\partial f}{\partial x_j} \frac{\partial \phi_{j-r+1}}{\partial x_i}(\epsilon, x_1, \ldots, x_{r-1}), \tag{15}$$

for each $i = 1, 2, \ldots, r-1$. However, since in a neighbourhood of (ϵ, x) eq. (12) still holds, we have that the real-valued differentiable function

$$\bar{f}(x_1', \ldots, x_{r-1}') = f\big(\epsilon, x_1', \ldots, x_{r-1}', \phi_1(\epsilon, x_1', \ldots, x_{r-1}'), \ldots, \\ \phi_{n-r+1}(\epsilon, x_1', \ldots, x_{r-1}')\big)$$

attains a local minimum at $x_1, x_2, \ldots, x_{r-1}$. Hence

$$0 = \frac{\partial \bar{f}}{\partial x_i}(x_1, \ldots, x_{r-1}) = \text{ right hand side of (15)},$$

(by chain rule) for each $i = 1, 2, \ldots, r-1$. It follows immediately that $\nabla f(\epsilon, x).v_i = 0$ for each $i = 1, 2, \ldots, r-1$; the result follows by Corollary 3.1.

5.5. *Proof of Corollary* 3.4

By part (i) of Corollary 3.2 we see that if (η, z) is in the $W^{\#}$ part of W^{+}, the $u_j(\eta, z) = 0$, $j = 2, 3, \ldots, m$. Further, if $(\eta, z) \in W^{+} \backslash W^{\#}$, then by part (ii) of Corollary 3.2

$$\nabla f(\eta, z).u_j(\eta, z) = 0, j = 2, 3, \ldots, m.$$

Hence in each neighbourhood Q_s, $W^{+} \cap Q_s$ is the zero set of a finite number of analytic functions. Hence W^{+} is a complex analytic variety.

In order to prove (ii), observe that if a minimum is attained at a singular point or a regular point at which the tangent space is vertical, then such a minimum is in W^{+}, by construction. If a minimum is attained at a regular point at which the tangent space is non-vertical, then by Corollary 3.3, $\nabla f(\epsilon, x) \in \widehat{T}^{\perp}(W_{\epsilon}, x)$ and hence (ϵ, x) is still in W^{+}.

5.6. *Proof of Proposition* 3.2

Let us show (i). Let $x \in \partial S_{\epsilon}$ be such that $(\epsilon, x) \in W^{\star}$. From the definition of $W^{+} = W$, observe that

$$\nabla f(\epsilon, x') \in \widehat{T}^{\perp}\big((W)_{\epsilon}, x'\big),$$

locally in the neighbourhood where $(W^{*})_{\epsilon}$ is a manifold of some fixed dimension r. But by Corollary 3.1, $\widehat{T}((W)_{\epsilon}, x')$ is spanned by a set of $(r-1)$ vectors. Without loss of generality we can assume that these are the vectors $v_1, v_2, \ldots, v_{r-1}$ from the proof of part (i) of Proposition 3.1. Hence, for $i = 1, 2, \ldots, v_{r-1}$

$$\nabla f(\epsilon, x').v_i = 0, \tag{16}$$

for all x' in that same neighbourhood. Since in that neighbourhood an analog of Eq. (12) also holds, we can think of a new function

$$\tilde{f}(\epsilon, x'_1, x'_2, \ldots, x'_{r-1}) = f(\epsilon, x'_1, \ldots, x'_{r-1}, \ldots, x'_n),$$

where x'_j's satisfy (12) for $j = r, \ldots, n$. By analogy with (15) we see that (16) implies that for each $i = 1, 2, \ldots, r-1$

$$\frac{\partial \tilde{f}}{\partial x_i}(\epsilon, x') = 0 \tag{17}$$

on the same neighbourhood. Thus, locally, $f(\epsilon, x')$ is a constant and therefore locally any point is a minimiser. This contradicts our assumption that x is a boundary point.

Item (ii) is an immediate consequence of (i).

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